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Abstract
The embedded ensembles were introduced by Mon and French (1975 *Ann. Phys., NY* **95** 90) as physically more plausible stochastic models of many-body systems governed by one- and two-body interactions than provided by standard random-matrix theory. We review several approaches aimed at determining the spectral density, the spectral fluctuation properties and the ergodic properties of these ensembles: moments methods, numerical simulations, the replica trick, the eigenvector decomposition of the matrix of second moments and supersymmetry, the binary correlation approximation, and the study of correlations between matrix elements.

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1. Introduction

Canonical random-matrix theory (RMT) as introduced by Wigner [38] considers ensembles of random matrices classified [16] by their symmetries. In the asymptotic limit of infinite matrix dimension, $N \rightarrow \infty$, this theory makes a number of remarkable predictions. The average spectrum (the spectral density) has the shape of a semicircle, the spectral fluctuations are universal (i.e. are under very weak conditions independent of the weight function used to define the ensemble, are parameter-free and dependent only on the symmetries of the ensemble), and the results are ergodic (so that for most members of the ensemble the spectral average of any observable coincides with the ensemble average of that same observable). For a recent review of the field, we refer to [22].

Nevertheless, early applications of canonical RMT to nuclear spectra encountered objections or, at least, questions. This is because in RMT, the number of independent random variables (i.e. of uncorrelated matrix elements) is of order N^2. Put differently, in RMT every state in Hilbert space is connected to every other such state by a matrix element which does not vanish and is assumed to be an independent random variable. But with the success

of the nuclear shell model, it had become clear that nuclei are governed by one- and two-body interactions (the mean field and the residual two-body interaction). In a representation of Hilbert space spanned by Slater determinants of single-particle wavefunctions (solutions of the mean-field equations), the residual two-body interaction has non-vanishing matrix elements only between determinants which differ in the occupation numbers of not more than two single-particle states, and most matrix elements of the residual two-body interaction, therefore, vanish. Even if one is prepared to accept a stochastic approach and considers the non-vanishing two-body matrix elements of the residual interaction as uncorrelated random variables, the ensuing stochastic model is very different from canonical RMT: the number of independent random variables is generically small compared to the dimension of a typical shell-model matrix. Why—so the question—can canonical RMT then serve as a model for the description of fluctuation properties of nuclear spectra?

French and Wong [18, 19] and Bohigas and Flores [9, 10] approached this question with the help of numerical simulations. These authors replaced the actual two-body matrix elements of the nuclear shell model by random variables (this, in essence, defines the two-body random ensemble (TBRE)) and studied the resulting spectral fluctuation properties. They concluded that the fluctuation properties were consistent with predictions of canonical RMT.

Because of the complexities of angular-momentum and spin coupling and the exclusion principle, the TBRE is not amenable to analytical investigations. Thus, the introduction of the k-body embedded ensembles (EGE(k)) by Mon and French [32] might be considered an essential step towards an analytical understanding of the numerical results obtained by French and Wong [18, 19] and Bohigas and Flores [9, 10] and, thereby, of the stochastic properties of nuclear spectra. Since then, the question has gained much wider significance: Canonical RMT has successfully been used [22] in such diverse many-body systems as atoms, molecules and quantum dots. All of these systems share with nuclei the feature that they are known to be governed essentially by effective one- and two-body forces. Thus, the embedded ensembles may be viewed as the generic models for stochasticity in many-body systems.

The embedded ensembles dispose of all the complexities of the couplings of angular momentum and spin while retaining the symmetries imposed by the exclusion principle. One considers m fermions in l degenerate single-particle states interacting via a random k-body interaction. The single-particle states carry no further quantum numbers such as spin or angular momentum. In contrast to shell-model calculations (where the single-particle states are usually not degenerate) degeneracy is here assumed in order to focus attention on the results of a stochastic k-body interaction. The interaction is not restricted to $k = 2$ (although this remains the physically most interesting case). This is done in order to obtain an understanding of the transition from the case $k = 2$ to the case $k = m$ which, as it turns out, is equivalent to canonical RMT. Just as in RMT, the embedded ensembles can be bestowed with unitary, orthogonal or symplectic symmetry, and one distinguishes accordingly EGUE(k), EGOE(k) and EGSE(k). As in canonical RMT, one is interested in universal properties. Therefore, one considers the limit of infinite matrix dimension. This limit is attained by letting the number of single-particle states go to infinity, $l \rightarrow \infty$. EGE(k) is, thus, a generic model for stochasticity in many-body systems governed by k-body interactions. Needless to say, EGE(k) can and has been generalized to the case of bosons.

The central questions in the theory of the embedded ensembles then are: (i) What is the shape of the spectral density? (ii) What are the spectral fluctuation properties? (iii) Are these properties universal? (iv) Are the spectra ergodic?

In this review, we present the status of the field in the light of these questions. We start with a definition of EGE(k). Then, we order the material by the analytical or numerical technique which has been used, rather than by the properties of EGE which have been addressed.

2. Definitions: the *k*-body embedded ensembles of fermions and bosons

We define the *k*-body embedded ensembles for fermions and for bosons for the cases of unitary and orthogonal symmetries. We do not consider symplectic symmetry throughout this paper.

We consider a set of l degenerate single-particle states labelled $|j\rangle$, with $j = 1, \ldots, l$. The associated creation and annihilation operators are denoted by $a_j^\dagger$ and a_j in the case of fermions and $b_j^\dagger$ and b_j in the case of bosons. These operators obey the usual (anti)commutation relations. Then, $|j\rangle = a_j^\dagger |0\rangle$ or $|j\rangle = b_j^\dagger |0\rangle$, with $|0\rangle$ the vacuum state.

To define the *k*-body interaction, it is useful to introduce operators which create a normalized state with $k \leqslant l$ fermions or k bosons from the vacuum. In the case of fermions, these are written as

$$\psi_{k,\alpha}^\dagger = \psi_{j_1, j_2, \ldots, j_k}^\dagger = \prod_{s=1}^{k} a_{j_s}^\dagger \tag{1}$$

where $j_1 < j_2 < \cdots < j_k$. The label k defines the number of single-particle creation operators, while α stands for the set of j. The corresponding annihilation operators are given by

$$\psi_{k,\alpha} = \left(\psi_{k,\alpha}^\dagger \right)^\dagger. \tag{2}$$

In the case of bosons, the *k*-body operators have the form

$$\chi_{k,\alpha}^\dagger = \chi_{j_1, j_2, \ldots, j_k}^\dagger = N_\alpha \prod_{s=1}^{k} b_{j_s}^\dagger \tag{3}$$

where $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_k$. Here, N_α is a normalization factor which guarantees that the state $\chi_{k,\alpha}^\dagger |0\rangle$ has norm unity. For every set of equal indices j containing n elements occurring in α, N_α contains the factor $1/\sqrt{n!}$.

The random *k*-body interaction for fermions is given by

$$V_k = \sum_{\alpha\gamma} v_{k;\alpha\gamma} \psi_{k,\alpha}^\dagger \psi_{k,\gamma}. \tag{4}$$

The coefficients $v_{k;\alpha\gamma}$ are random variables with a Gaussian probability distribution, mean value zero and a common second moment. The V_k form an ensemble of random operators. In the case of unitary symmetry, the $v_{k;\alpha\gamma}$ are complex. The only non-vanishing second moment is

$$\overline{v_{k;\alpha\gamma} v_{k;\alpha'\gamma'}^*} = v^2 \delta_{\alpha\alpha'} \delta_{\gamma\gamma'}. \tag{5}$$

The bar denotes the ensemble average. The Kronecker deltas stand for the string $\delta_{j_1 j_1'} \delta_{j_2 j_2'} \ldots$. In the case of orthogonal symmetry, the $v_{k;\alpha\gamma}$ are real and obey

$$\overline{v_{k;\alpha\gamma} v_{k;\alpha'\gamma'}} = v^2 [\delta_{\alpha\alpha'} \delta_{\gamma\gamma'} + \delta_{\alpha\gamma'} \delta_{\gamma\alpha'}]. \tag{6}$$

Following the usage in canonical RMT, we distinguish both cases by writing $V_k(\beta)$ with $\beta = 1, 2$ for orthogonal and unitary symmetries, respectively. The *k*-body random interaction for bosons is defined in complete analogy with equation (4). The parameter v^2 sets the scale of the energy. As long as we consider only a single EGE(k), we can put $v^2 = 1$ without loss of generality.

We consider a Hilbert space obtained by distributing $m \leqslant l$ fermions or m bosons over the single-particle states $|j\rangle$. A complete set of basis states for fermions (bosons) is given by $\psi_{m,\alpha}^\dagger |0\rangle$ (by $\chi_{m,\alpha}^\dagger |0\rangle$, respectively). The dimensions of these two spaces are

$$N = \binom{l}{m} \qquad \text{for fermions}$$

$$N_B = \binom{l+m-1}{m} \qquad \text{for bosons.}$$

$\hspace{12cm}$ (7)

Here and in the following, we use the index B for bosons while we typically work without such an index for the more frequent case of fermions. We take $m \geqslant k$ and consider the matrix of the k-body interaction V_k in that m-particle Hilbert space. The matrix elements $\langle \mu | V_k | \nu \rangle$ have the form

$$\langle \mu | V_k | \nu \rangle = \langle 0 | \psi_{m,\mu} V_k \psi_{m,\nu}^\dagger | 0 \rangle \qquad (8)$$

and analogously for bosons. Using equation (4) for V_k, it is easily seen that for $m = k$, the matrices of the k-body interaction V_k coincide with the GUE or the GOE, depending on the symmetry chosen, both for fermions and for bosons. This is not the case for $m > k$. In this case, we speak of the k-body embedded Gaussian ensembles (EGE(k)) of random matrices in the m-particle Hilbert space. We use the notation EGUE(k) and EGOE(k) for the cases of unitary and orthogonal symmetry, respectively. Aside from the (unitary or orthogonal) symmetry, the embedded ensembles are defined in terms of the three parameters l, m, k. We recall that in canonical RMT, the limit $N \to \infty$ of infinite matrix dimension yields universal results. For the EGE(k), we proceed likewise by taking the limit $l \to \infty$ which implies the limit of infinite matrix dimension. For fermions, we have the obvious constraint that $k \leqslant m \leqslant l$. The limit $l \to \infty$ with fixed k can be attained by keeping m fixed, or by keeping the ratio $m/l \leqslant 1$ fixed. Brody *et al* [14] define the dilute limit by taking $l \to \infty, m \to \infty, m/l \to 0$. For bosons, m may be larger than l. Since EGE(k) is constructed in such a way that EGE(m) coincides with canonical RMT for both fermions and bosons, and since in the dilute limit the distinction between fermions and bosons becomes irrelevant, we expect that the spectral properties of EGE(k) do not differ qualitatively for fermions and bosons except for the case where the number of m bosons is close to or exceeds l, the dimension of the underlying single-particle space. This is indeed what is found.

Canonical RMT is invariant under unitary or orthogonal transformations of Hilbert space. While this statement carries over trivially to the k-body ensembles embedded into a Hilbert space spanned by k particles, that invariance is lost for $m > k$. The calculation of spectral properties of canonical RMT greatly benefits from such an invariance property. Thus, the determination of the spectral properties of the embedded ensembles is much more difficult than for canonical RMT.

3. Moments methods

The first analytical approach to EGE(k), developed by Mon and French [32], calculates moments of the matrix elements of V_k and related quantities. It has led to insights into the shape of the EGE(k) spectra (the spectral density) and to the development of statistical spectroscopy in nuclei, and has given a limited answer to the question of ergodicity. These topics are reviewed in [14, 28]. The approach has mainly been used for fermions in the context of the nuclear shell model, although it can also be applied for bosons as in the interacting boson model for nuclei. We confine ourselves to fermions.

In a Hilbert space with m fermions, the pth moment of V_k is defined as $M_p(m) = (1/N)\,\mathrm{tr}\overline{[(V_k)^p]}$. It is obvious that all odd moments vanish. The even moments can be evaluated using Wick contraction of the V_ks. The moment is then the sum over all patterns of pairwise contractions. Assigning the same letter to pairs of Wick-contracted V_k, the fourth

moment is, for instance, given by $M_4(m) = (1/N) \operatorname{tr}(2A_k A_k B_k B_k + A_k B_k A_k B_k)$ where we have used the cyclic invariance of the trace. For $l \to \infty$, the low moments can be worked out using combinatorial arguments [32]. Knowledge of the moments yields information on the average spectral density $\overline{\rho(E)}$. This is because of the relation

$$\operatorname{tr}\overline{(V_k)^p} = \overline{\sum_i (E_i)^p} = \int \mathrm{d}E \overline{\rho(E)} E^p \tag{9}$$

where the E_i are the eigenvalues of V_k.

Canonical RMT is obtained for $k = m$ and predicts the spectral density to have a semicircular shape. On the other hand, in the dilute limit, all Wick-contracted pairs of V_k become independent of each other, the moment $M_{2p}(m)$ is given by $M_{2p}(m) = (2p - 1)!!(M_2(m))^p$, and the spectral density is Gaussian. This shows that for fixed k and for m monotonically increasing from $m = k$, the spectral density undergoes a transition from a semicircular to a Gaussian shape. For $k = 1$, the transition can be worked out easily because the level density for $m > 1$ is the convolution of m semicircles. Already for $m = 9$, the spectral shape is very close to Gaussian. This argument fails for $k \geqslant 2$. In [32], the even moments $M_{2p}(m)$ are worked out for $p = 1, 2, 3, 4$. The values obtained are consistent with the gradual transition from semicircular to the Gaussian shape of the spectral density.

In nuclear theory, sums over all final states of the strength of some transition operator $\mathcal{O}$ play an important role. Examples are single-particle transfer strengths or the Gamow–Teller transition strength. Such transition strength sums can be worked out too using the moments method. The reason is that $K = \mathcal{O}^\dagger \mathcal{O}$ can be expressed in terms of creation and annihilation operators, and terms such as $\operatorname{tr}\overline{[(V_k)^p K]}$ can again be calculated using combinatorial arguments, especially in the dilute limit. This method has been very successfully applied in the framework of the nuclear shell model. In many cases, EGOE(k) results obtained in this way show remarkable agreement with results of detailed shell-model calculations. We do not dwell on this important issue because an extensive review has appeared quite recently [28].

Using the moments method to calculate the spectral density or transition strength sums, we always use the ensemble average. Is it justified to compare such averages with properties of a given physical system (as we do when comparing with shell-model calculations)? While a given physical system obviously does not permit averaging over the (physically non-existent and fictitious) ensemble, it does permit the determination of spectral averages. For instance, by grouping neighbouring levels into bins, we can determine the mean spectral density. This poses the question: are spectral averages (taken over a specific system) and ensemble averages (over the EGE(k)) identical? This problem is referred to as the problem of ergodicity, in analogy to the well-known ergodic problem in classical statistical mechanics concerning the equality of phase-space average and the long-time average over a single trajectory.

For ensemble averaging to be relevant for an individual system, the property under study must vary little from member to member of the ensemble. In the context of the moments method, the question of ergodicity is answered by calculating the variances

$$\Sigma_{p,q}^{(2)}(m) = \overline{(1/N) \operatorname{tr}(V_k^p)(1/N) \operatorname{tr}(V_k^q)} - \overline{(1/N) \operatorname{tr}(V_k^p)} \; \overline{(1/N) \operatorname{tr}(V_k^q)}. \tag{10}$$

For canonical random-matrix theory and in a normalization where the individual moments are finite in the limit $N \to \infty$, the variances $\Sigma_{p,q}^{(2)}$ vanish asymptotically: The canonical ensembles are ergodic as far as the spectral density is concerned. For the embedded ensembles, the calculation of the variances in equation (10) involves contractions between pairs of operators located in different traces as well as in the same trace. The treatment of the first kind of pairs requires combinatorial techniques beyond those developed for the moments [32]. Mon and French could show that for EGOE(k), $\Sigma_{p,q}^{(2)}(m)$ vanishes asymptotically for $l \to \infty$. Thus, they

could demonstrate ergodicity for the spectral density. Their argument can easily be extended to EGUE(k).

Unfortunately, the moments method, while very useful for the spectral density and for statistical spectroscopy, is only of limited use for calculating spectral fluctuations. For canonical RMT, Mon and French consider density fluctuations as excitations of the semicircle and use this approach to study spectral GOE fluctuations. They succeed in re-deriving the logarithmic dependence of the spectral stiffness on the length of the energy interval. However, the method apparently does not permit the study of short-scale fluctuations such as the nearest-neighbour spacing distribution. Mon and French apply the same method to EGOE(k) with $m \gg k$. They show that long-range fluctuations of the spectrum are large. As a consequence, neither the position of a given (numbered) eigenvalue nor the position of the centre of the spectrum or the variance are ergodic. We return to this point in section 6. The short-range fluctuations, however, which yield information about spectral statistics, pose difficulties. To quote from [14]: '... there is no real theory yet for EGOE fluctuations, the gap being one of the most significant ones in the entire subject.'.

4. Numerical results

Insight into the spectral fluctuation properties of the embedded ensembles may be gained from numerical simulations. Such simulations have been performed from the early days of the field. Here we review separately the simulations for fermionic and for bosonic systems.

4.1. Fermionic systems

Prior to the introduction of EGE(k) by Mon and French [32], numerical simulations used the two-body random ensemble (TBRE). This ensemble was introduced in the context of nuclear physics where standard shell-model techniques were available to calculate the spectra of given spin and parity (and isospin if relevant) in terms of a fixed two-body interaction. The latter was replaced by an ensemble of two-body matrix elements with a Gaussian distribution, zero mean value and a common second moment. The antisymmetrized (fermionic) random two-body matrix elements thus define the residual interaction among m particles distributed over l degenerate single-particle states. Inasmuch as nuclei are governed by one- and two-body forces, the TBRE is clearly a more suitable stochastic model than canonical RMT. The complexities arising from the angular-momentum and spin couplings, from correlations between many-body matrix elements induced by the two-body character of the interaction, and from Pauli's exclusion principle made it impossible to treat the TBRE analytically. This led French and Wong [18, 19] and Bohigas and Flores [9, 10] to resort to numerical simulations using matrices of rather small dimensions ($\leqslant 61$). The main question addressed in these simulations was: are the spectral fluctuation properties of the TBRE the same as those of the GOE?

The main results of these early works are: (i) The spectral density of the TBRE is Gaussian, in agreement with the results of shell-model calculations with realistic interactions. (ii) The spectral density displays a transition from Gaussian to semicircular shape as the rank of the interaction k is increased. (iii) Unfolding the spectrum of each member of the ensemble separately yields good agreement of the fluctuation properties of the TBRE with those predicted by canonical RMT. (iv) The fluctuations of the position of the first eigenvalue of the TBRE are stronger than those for the GOE. (v) The TBRE is neither stationary nor ergodic. These results hold also for simplified versions of the TBRE, which do not involve, or at least do so only partially, the complexity of the angular momentum algebra.

Early results on the *p*th nearest-neighbour spacing distribution displayed systematic deviations between the TBRE and the GOE [11] while experimental data showed agreement with GOE predictions. This seemed to suggest the existence of an effective many-body interaction in nuclei. It was soon realized, however, that the deviations were related to the method of analysis of the spectra and caused by the non-ergodic properties of the TBRE [39, 20]. More precisely, the deviations were due to the difference between ensemble averaging and spectral averaging. Ensemble averaging uses a single staircase function (the integrated level density) to unfold the spectra of all members of the ensemble. This function is obtained by superposing the levels of all members of the ensemble. Early numerical results [18, 9] (and later analytical studies [21, 32]) had shown that in the dilute limit ($l \gg m \gg k, l \to \infty, m \to \infty$ with $m/l \to 0$), the mean level density is Gaussian. This spectral shape was accordingly used for ensemble unfolding. On the other hand, spectral averaging uses a separate staircase function for each member of the ensemble. It was found that the deviations from GOE behaviour obtained by ensemble averaging were related to the variance of the spacing distribution; an analytical estimate of this effect, linking ensemble and spectral average, was also obtained [20, 12]. A first systematic comparison of spectral and (a corrected) ensemble averaging, using matrices of rather small dimensions, was carried out in [13]. This comparison showed remarkable agreement between the TBRE results and GOE predictions. The agreement invalidated the earlier conclusion that nuclear interactions are mainly of many-body type, and has left open the question of the rank of the effective nuclear interaction [12, 14].

Unfolding the spectrum is technically a subtle task. It can considerably influence the results on spectral fluctuations. Attention was therefore devoted to a consistent way of constructing ensemble averages which yield correct spectral fluctuations [14, 29]. This problem has recently again attracted attention [17, 23]. In the dilute limit, the spectral density is Gaussian. However, knowledge of this fact is not sufficient for a proper unfolding of the spectra of the EGE(*k*) and of the TBRE as obtained in numerical simulations with *finite-dimensional* matrices. This is because for such matrices the first and second moments of the spectrum are not ergodic (cf section 6.3). Hence, these moments must be normalized for every member of the ensemble. The sensitivity of the spectral fluctuations to these corrections was addressed in [17], where either the first or the second moment, or both, are corrected for every matrix of the ensemble. The results show that, in the centre of the spectrum, the nearest-neighbour spacing distribution is not sensitive to such corrections. For the two-point function, ensemble unfolding with the Gaussian mean level density yields results that deviate clearly from predictions of canonical RMT. A slight correction results from adjusting the width of the distribution for every member of the ensemble. A drastic improvement is obtained when the distribution for every member of the ensemble is recentred. Applying both corrections together yields results which are hardly distinguishable from those of canonical RMT. Once the low moments of the spectral distributions are adjusted so as to coincide for all members of the ensemble, a polynomial fit for the ensemble staircase function is usually implemented. This procedure yields a function with which the spectra can be meaningfully unfolded.

A method for spectral unfolding which is somewhat more complicated but has a firm theoretical basis [32, 14] for the embedded ensembles, uses a Gram–Charlier expansion. Starting from the fact that in the dilute limit the spectral density is Gaussian, one writes the spectral density $\rho_s(x)$ of an ensemble of matrices of finite dimension in the form [14, 29]

$$\rho_s(x) = \rho_G(x)\left[1 + \sum_{n \geqslant 3} \frac{c_n H_n(x)}{n!}\right]. \tag{11}$$

Here, $x = (E - E_c)/\sigma$ is the recentred energy in units of its standard deviation, ρ_G is the Gaussian distribution and $H_n(x)$ is the Hermite polynomial of degree n. The Hermite

polynomials describe the long-wavelength oscillations of $\rho_s(x)$ (long in terms of the mean level spacing) about the asymptotic value $\rho_G(x)$. The coefficients c_n are adjusted so as to minimize Δ_{RMS}, the overall root-mean-square error of the level-to-level deviations of the staircase function calculated from equation (11) from the data. Explicit expressions for c_3 and c_4 are given in [29]. One uses equation (11) with some maximum value n_0 in the sum over n. This value should be sufficient to eliminate any secular trend in Δ_{RMS} while preserving the fluctuations of the spectrum. The value of n_0 is usually determined by calculating Δ_{RMS} as a function of n_0, and by choosing the smallest value beyond which no significant corrections to Δ_{RMS} are found. For the EGE(k) this method yields good agreement with the spectral fluctuation properties predicted by canonical RMT [14, 29, 28].

The spectral fluctuation properties at the edge of the spectrum have also received increased attention. These are interesting for a comparison of TBRE and EGOE(k) results with the positions of low-lying nuclear states. Bohigas and Flores [10] compared the properties of the low-lying part of the spectrum of the TBRE and the GOE. They showed that the widths of the positions of individual eigenvalues were much larger for the TBRE than for the GOE. Cota *et al* [15] fitted the nearest-neighbour spacing distribution to a Brody distribution, $P(s, \omega) = \alpha(\omega + 1)s^\omega \exp[-\alpha s^{\omega+1}]$ with $\alpha = \{\Gamma[(\omega+2)/(\omega+1)]\}^{\omega+1}$, obtaining for the Brody parameter the value $\omega \approx 0.80 \pm 0.05$. More recent results by Flores *et al* [17] show that the semi-Poisson distribution, $P(s) = 4s \exp[-2s]$, gives a better fit than the Brody distribution, if the levels of each member of the ensemble are normalized first according to an individual Gaussian distribution (spectral unfolding).

Numerical simulations using matrices of dimension $\sim$3000 have repeatedly found agreement between the fluctuation properties of the EGE(k) and the TBRE at the centre of the spectrum on the one hand, and the predictions of canonical RMT on the other. Therefore, it is often taken for granted that a random $k = 2$ part in the Hamiltonian is sufficient to induce level fluctuations of canonical RMT type. This point of view is at odds with the fact that many-body systems with random few-body interactions (both for fermions and bosons) may possess a high degree of order in the low-lying part of the spectrum [24, 6, 25, 7]. For instance, for the TBRE with all the angular momentum couplings taken into account explicitly, it was found that there is a statistical preference for $J^\pi = 0^+$ ground states despite the fact that these states account only for a small fraction of Hilbert space [24]. Moreover, these $J^\pi = 0^+$ ground states are separated by a gap from the lowest excited state. Other properties such as localization in Fock space [26] and odd–even binding effects [33] show that pairing effects are robust properties of two-body interactions. Such properties are usually understood in Hamiltonians which involve some collective behaviour. All these results, based on Hamiltonians with random interactions, contradict the basic philosophy of and the predictions based on canonical RMT and suggest that the embedded ensembles may not always yield results which coincide with canonical RMT.

4.2. Bosonic systems

Early analytical results by Kota and Potbhare [27] indicated a Gaussian spectral density both in the dilute and the dense limits. The dense limit which exists only for bosons, is defined as $m \to \infty$ for fixed l and k. Manfredi [30], using matrices of dimension 364 ($l = 4, m = 11$), compared for the first time the spectral fluctuation properties at the centre of the spectrum of EGE(k) for bosons with the results of canonical RMT. Using spectral unfolding he concluded that there is no significant difference between the two ensembles. Patel *et al* [34] considered this problem in the dense limit, studying the case of $l = 5$ and $m = 10$. These authors constructed the ensemble-averaged staircase function using a sixth-order Gram–Charlier expansion (11).
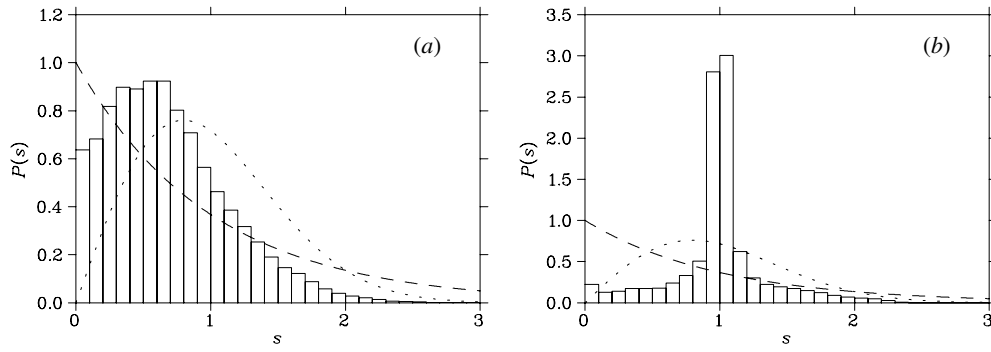
Figure 1. Nearest-neighbour spacing distribution $P(s)$ obtained by (*a*) ensemble unfolding and (*b*) spectral unfolding. The dotted curves correspond to the Wigner surmise and the dashed to the Poisson distribution. We note the different vertical scales used in the frames. Taken from [2].
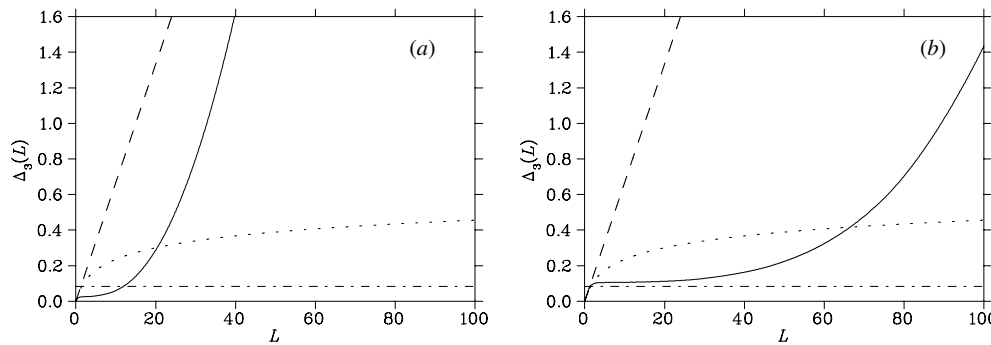
Figure 2. Δ_3-statistics (solid lines) measured at the centre of the spectrum after (*a*) ensemble unfolding and (*b*) spectral unfolding. The dotted curves correspond to the GOE results, the dashed to a Poissonian spectrum and the dotted-dashed to a picket-fence spectrum. Taken from [2].

They found excellent agreement with canonical RMT results for the nearest-neighbour spacing distribution (with a Brody parameter $\omega = 0.95$) and for the Δ_3-statistic. These results led them to conclude that the embedded ensembles possess generically (for fermions and bosons) the spectral fluctuation properties of canonical RMT.

Recently it was found analytically, however, that the embedded ensembles for bosons are not ergodic in the dense limit [1, 2]. More precisely, the fluctuations of the centroids and widths of the spectrum (in units of the average width) do not vanish in this limit, but attain constant values (cf subsection 6.3). This fact implies that unfolding the spectra by ensemble averaging or by spectral averaging will yield different results. The Hilbert-space dimension for bosons is given by equation (7) and it is then clear that the effect can be best displayed numerically for $l = 2$.

Figures 1 and 2 illustrate the non-ergodic behaviour for $m = 3000$ bosons and the case of a two-body interaction (with 1512 members forming the ensemble) for the nearest-neighbour spacing distribution and for the Δ_3-statistic, both taken in the centre of the spectrum. Spectral unfolding (ensemble unfolding) was done by fitting a polynomial of degree 11 to the staircase function of each member of the ensemble (to the ensemble-averaged staircase function, respectively). For ensemble unfolding, the individual spectra were not recentred or rescaled since there is now no theoretical support for this procedure. In fact, rescaling and

recentring the spectra would yield a non-Gaussian average level density. Interestingly, in the case of spectral unfolding the nearest-neighbour spacing distribution is dominated by a large peak centred at $s = 1$. This suggests that individual spectra have an almost constant level spacing, i.e. are close to spectra of picket-fence type. This is further supported by the large plateau observed in the plot for the Δ_3-statistic.

For the same case ($l = 2, m = 3000$) the structure of the eigenfunctions was also investigated [1, 2]. It was found that individual spectra consist generically of a superposition of independent sequences of levels with constant spacings as if there was some symmetry present in the system. When a new sequence appears, the old and the new sequences are interlaced, and the level spacing is no longer constant for a while. In the staircase function, this is seen as a sudden change in the average slope, i.e. as a kink. The first few eigenstates of the new sequence are localized in Hilbert space, even though they may correspond to rather highly excited states of the spectrum. This is illustrated in figure 3, where the squares of the coefficients c_{in} of the eigenvector expansion in an ordered many-body basis $|\mu_n\rangle$ are plotted (cf [2]).

These numerical results show that in the dense limit, the bosonic EGE(k) differ significantly from the canonical ensembles of RMT. In the case $l = 2$ it has been established that the difference is due to the exact integrability of the system in the semiclassical limit [5]. Essentially, this case corresponds to the quantization of a classical two-degrees of freedom system which possesses two independent integrals of motion, the energy and the number of particles. For arbitrary values of l, the corresponding classical system has l degrees of freedom. As l increases, chaotic trajectories appear in phase space, typically for energies around the centre of the spectrum, and there is a corresponding transition in the spectral fluctuations of the quantum system.

In concluding this section we mention that the spectral properties at the edge of the spectrum have also been investigated for bosons. Using a random interaction in the framework of the interacting boson model, Bijker and Frank have demonstrated the emergence of collective motion in the low-lying part of the spectrum, and have studied the dependence of collective motion on the rank k of the interaction [6–8]. When the number of interacting bosons m is large enough with respect to k, they find a preponderance of $J^\pi = 0^+$ ground states as well as definite evidence for the appearance of vibrational and rotational structures. For $m \sim k$ they do not obtain an indication of vibrational or rotational band structures.

5. The method of replicated variables

Verbaarschot and Zirnbauer [37] used the replica trick developed in statistical mechanics for the study of spin glasses and of Anderson localization to investigate spectral properties of the GOE and EGOE(k). Using the replica trick, one writes the observable under study as the logarithmic derivative of a suitably chosen generating function Z, replaces $\log Z$ by $\lim_{n\to 0}[Z^n - 1]$, evaluates $\overline{Z^n}$ for positive integer values of n and takes the limit $n \to 0$ of the resulting expression. Similar to what happens in the supersymmetry approach described in section 6 below, averaging Z^n yields in the exponent of Z^n a quartic term in the integration variables which contains the matrix $A^{(k)}_{\mu\nu;\rho\sigma}(\beta)$ defined in equation (12). The authors introduce matrix elements of the operators $\psi^\dagger_{k,\alpha}$ which connect the m-particle states with the $(m - k)$-particle states and perform the Hubbard–Stratonovich transformation in this mixed representation.

They use the saddle-point approximation for the one-point function and find a semicircle for the spectral density. The moments method having shown that the semicircle does not apply for all values of (m, k), points to the need to study the loop expansion. It is shown that there
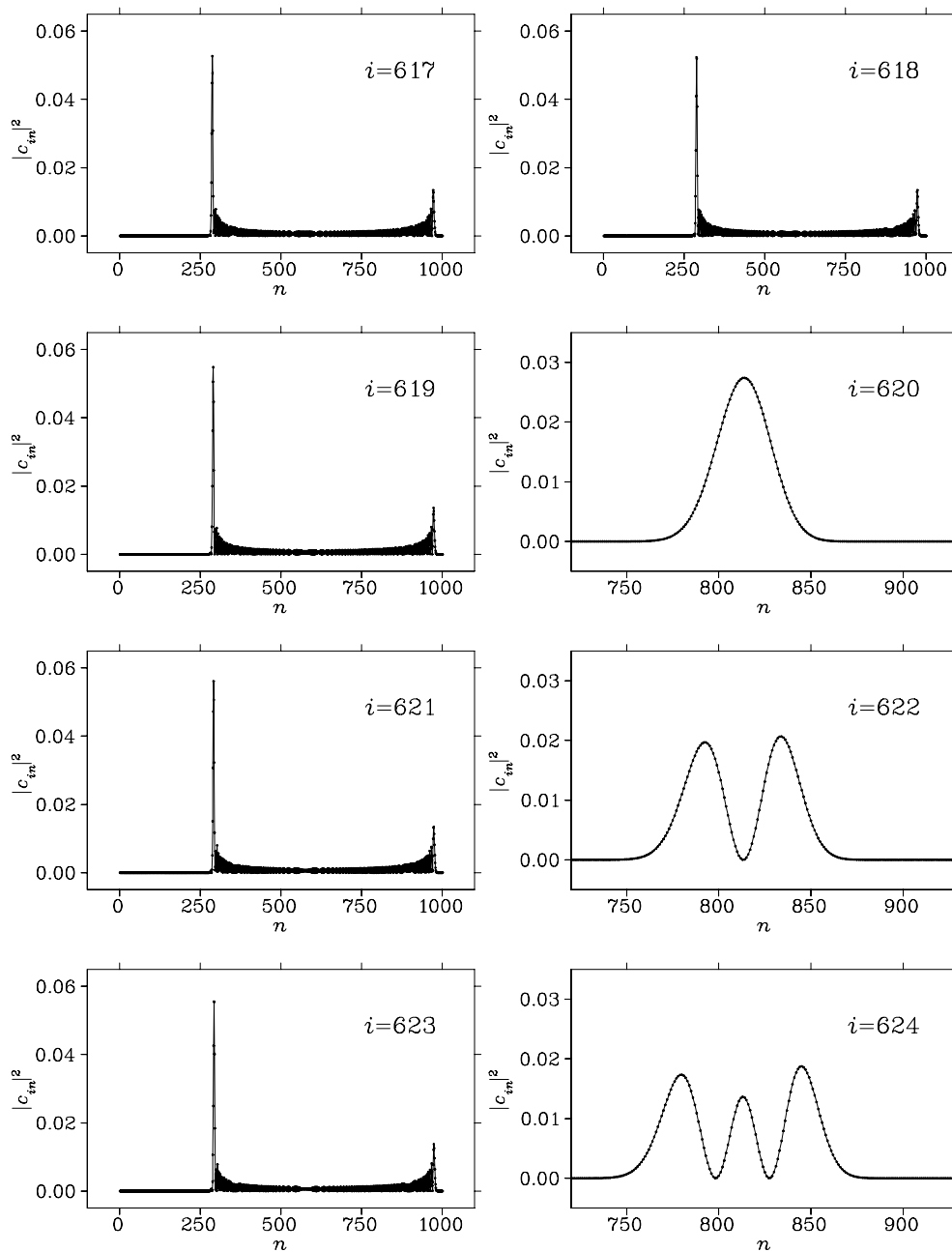
Figure 3. Intensities $|c_{in}|^2$ of a sequence of eigenvectors in the many-body basis in the vicinity of a kink in the staircase function. The two overlapping segments of nearly equidistant levels are easily distinguished by the structure of the corresponding eigenfunctions. Taken from [2].

are terms in the loop expansion which correct the semicircle. However, it is not possible to evaluate all such corrections.

The saddle-point approximation is then applied to the two-point function within the same mixed representation as used for the one-point function. The saddle-point solution allows

for the existence of a Goldstone mode. This mode carries the variable $s = (E_1 - E_2)/d$ where d is the mean level spacing. It is with respect to this variable that fluctuation properties of the spectrum are measured. The two-point function is evaluated using an expansion in inverse powers of s, the point being that the stiffness of the GOE spectrum relates to the occurrence of the factor s^{-2} in the average two-point function. Difficulties and, in fact, uncertainties arise from the need to integrate over the massive (i.e. the non-Goldstone) modes. With this proviso, it is found that the leading terms in the loop expansion combine to yield the value $-(17/2)(1/(\pi^2 s^2))$ for the density–density correlation function. This result has to be compared with the GOE value $-(1/(\pi^2 s^2))$. Led by early numerical results reviewed in section 4, the authors believe that they should have obtained the GOE result and speculate that the difference is due to high-order terms in the loop expansion. They show that if the density–density correlation function is $\propto s^{-2}$ then stiffness of the spectrum is implied.

Results obtained more recently [1–4] and reviewed in sections 6 and 7 cast some doubt on the assertion that the spectral fluctuation properties of EGE(k) coincide, for all values of k, with those of the canonical ensembles, at least in the limit $l \to \infty$. This would imply that higher-order loop corrections in the approach of [37] could modify the factor $(17/2)$ in a k-dependent fashion.

6. The matrix of second moments

A novel approach to determining spectral properties of EGE(k) was developed in [1–4]. We describe this approach here for the case of fermions. The case of bosons can be treated in complete analogy. The starting point is the observation that the matrix elements $\langle \mu | V_k | \nu \rangle$ of the stochastic operator V_k are themselves also random variables and have a Gaussian probability distribution with mean value zero. Therefore, all properties of the embedded ensembles are completely determined by the matrix $A^{(k)}_{\mu\nu;\rho\sigma}(\beta)$ of second moments defined by

$$A^{(k)}_{\mu\nu;\rho\sigma}(\beta) = \overline{\langle \mu | V_k(\beta) | \sigma \rangle \langle \rho | V_k(\beta) | \nu \rangle}. \tag{12}$$

The idea is to extract information on EGE(k) by studying the properties of the matrix $A^{(k)}_{\mu\nu;\rho\sigma}(\beta)$. Performing the average yields

$$A^{(k)}_{\mu\nu;\rho\sigma}(\beta) = \sum_{\alpha\delta} \left[\langle \mu | \psi_\alpha^\dagger \psi_\delta | \sigma \rangle \left[\langle \rho | \psi_\delta^\dagger \psi_\alpha | \nu \rangle + \delta_{\beta 1} \langle \rho | \psi_\alpha^\dagger \psi_\delta | \nu \rangle \right] \right]$$

$$= \sum_{\alpha\delta} \left[\langle \mu | \psi_\alpha^\dagger \psi_\delta | \sigma \rangle \left[\langle \rho | \psi_\delta^\dagger \psi_\alpha | \nu \rangle + \delta_{\beta 1} \langle \nu | \psi_\delta^\dagger \psi_\alpha | \rho \rangle \right] \right]. \tag{13}$$

In the last of equations (13) we have used the reality of the matrix element $\langle \nu | \psi_\delta^\dagger \psi_\alpha | \rho \rangle$.

6.1. Duality. Eigenvector expansion

In the unitary case, there is a connection between the matrices $A^{(k)}(2)$ for the k-body interaction and $A^{(m-k)}(2)$ for the $(m - k)$-body interaction. The relation is referred to as duality and reads

$$A^{(k)}_{\mu\nu;\rho\sigma}(2) = A^{(m-k)}_{\mu\sigma;\rho\nu}(2). \tag{14}$$

We note the difference in the sequence of indices $\{\mu\nu\rho\sigma\}$ on the two sides of this equation. The proof of equation (14) rests on the fact that for every m-fermion state $|\mu\rangle$ and for every operator $\psi_{k,\alpha}$ with $\psi_{k,\alpha}|\mu\rangle \neq 0$, there exists a uniquely defined operator $\psi_{(m-k),\gamma}$ such that $\psi_{(m-k),\gamma}\psi_{k,\alpha}|\mu\rangle = |0\rangle$, the vacuum state. We emphasize that the duality relation has nothing to do with particle–hole symmetry. It applies likewise to bosons and can be extended to the

case $\beta = 1$. The duality relation obviously assigns a special role to the case $2k = m$. This is the reason why spectral properties of EGE(k) change at $2k = m$.

For $2m > l$, the available single-particle states are more than half filled, and it is tempting to use particle–hole symmetry to simplify the algebra. Rewriting V_k in terms of hole operators and bringing the hole creation operators up front, one creates a sum of k'-body interactions with $k' = 1, 2, \ldots, k$. This is why particle–hole symmetry is of limited use only, except for the dilute limit.

A very useful tool in the analysis of spectral properties of EGE(k) is the eigenvector expansion of the matrix $A^{(k)}$. We take the unitary case, $\beta = 2$, for fermions and consider the matrix $A^{(k)}$ of second moments as a matrix in the product space $\{\mu\nu\}$. In this space, $A^{(k)}(2)$ is a Hermitean matrix and can, therefore, be diagonalized. The eigenvalue equation reads

$$\sum_{\rho\sigma} A^{(k)}_{\mu\nu;\rho\sigma}(2) C^{(sa)}_{\sigma\rho} = \Lambda^{(s)}(k) C^{(sa)}_{\mu\nu}. \tag{15}$$

The index $s = 0, 1, \ldots$ labels different eigenvalues $\Lambda^{(s)}$ and the index a labels degenerate eigenvectors. The eigenvectors are orthogonal. We choose the normalization

$$\sum_{\mu\nu} C^{(sa)}_{\mu\nu} C^{(tb)}_{\nu\mu} = N\delta_{st}\delta_{ab}. \tag{16}$$

Provided that the eigenvectors form a complete set, the matrix $A^{(k)}$ can be expanded in the form

$$A^{(k)}_{\mu\nu;\rho\sigma}(2) = \frac{1}{N} \sum_{s=0}^{m-k} \sum_a \Lambda^{(s)}(k) C^{(sa)}_{\mu\nu} C^{(sa)}_{\rho\sigma}. \tag{17}$$

In writing equation (17) we have anticipated the fact that for $s > (m-k)$, the eigenvalues $\Lambda^{(s)}$ vanish. Knowledge of the expansion (17), i.e. of the eigenvectors and eigenvalues of $A^{(k)}$, makes it possible to calculate the low moments of V_k. Moreover, using the expansion (17) in the framework of supersymmetry allows for the use of the Hubbard–Stratonovich transformation. The orthogonal case is treated analogously. Combining the results for $\beta = 1, 2$, we obtain

$$A^{(k)}_{\mu\nu;\rho\sigma}(\beta) = \frac{1}{N} \sum_{s=0}^{m-k} \sum_a \Lambda^{(s)}(k) \left[C^{(sa)}_{\mu\nu} C^{(sa)}_{\rho\sigma} + \delta_{\beta 1} C^{(sa)}_{\mu\rho} C^{(sa)}_{\nu\sigma} \right]. \tag{18}$$

To appreciate the significance of the eigenvector expansion equation (17), it is useful to recall the form of the second moment of the GUE Hamiltonian $H_{\mu\nu}$,

$$\overline{H_{\mu\sigma} H_{\rho\nu}} = \frac{\lambda^2}{N} \delta_{\mu\nu}\delta_{\rho\sigma}. \tag{19}$$

We have put $v^2 = 1$ for EGE(k) and, by analogy, put $\lambda = 1$ for the GUE. The fact that GUE coincides with EGUE(m) then implies that for $k = m$, equation (17) reduces to equation (19). Put differently, the one non-vanishing eigenvalue of the GUE is $\Lambda^{(0)}(m)$, and the associated non-degenerate eigenvector is $C^{(0)}_{\mu\nu} = \delta_{\mu\nu}$. All other orthogonal eigenvectors belong to eigenvalue zero. The comparison shows that the eigenvalue expansion equation (17) is the natural generalization of equation (19) for the second moment of the GUE Hamiltonian to EGUE(k). The Kronecker deltas in equation (19) express the unitary invariance of GUE. The fact that for $k < m$ the sum over s in equation (17) extends up to $(m-k)$, with eigenvectors $C^{(sa)}_{\mu\nu}$ which differ from Kronecker deltas, is due to the fact that V_k does not possess this unitary invariance for $k < m$. We remark in parenthesis that the eigenvalue in equation (19) obviously has value unity, while it turns out that for $m = k$, the eigenvalue $\Lambda^{(0)}(m)$ in equation (17) has the value $\binom{l}{k}$. The difference is due to the fact that when we use second quantization to write

GUE $= V_m$ and work out $A^{(m)}(2)$, we have to count the number of holes which is $\binom{l}{k}$. In the appendix of [37], results closely related to equations (12) to (18) were derived.

The form of the eigenvectors and eigenvalues remains to be determined. The eigenvectors $C_{\mu\nu}^{(sa)}$ have the form

$$C_{\mu\nu}^{(sa)} = \langle\mu|\psi_{s,\alpha}^{\dagger}\psi_{s,\gamma}|\nu\rangle, \tag{20}$$

where a stands for the set (α, γ). The form equation (20) applies for all $s = 0, 1, \ldots, m$ independently of the actual value of the eigenvalue $\Lambda^{(s)}(k)$. Whenever two or more single-particle indices in the sets α and γ coincide, special attention is required [4]. For this reason, the number $D^{(s)}$ of orthonormal degenerate eigenvectors $C_{\mu\nu}^{(sa)}$ belonging to fixed s (and, thus, the dimension of the corresponding subspace of Hilbert space) are given by

$$D^{(s)} = \binom{l}{s}^2 - \binom{l}{s-1}^2. \tag{21}$$

It follows that $\sum_{s=0}^{m} D^{(s)} = \binom{l}{m}^2$, the dimension of the product space $\{\mu\nu\}$, so that the eigenvectors form a complete set. The eigenvalues are given by

$$\Lambda^{(s)}(k) = \binom{m-s}{k}\binom{l-m+k-s}{k}. \tag{22}$$

We observe that $\Lambda^{(s)}(k) = 0$ for $s > (m-k)$. This fact limits the sums over s in equations (17) and (18). Using the eigenvector expansion in the duality relation, equation (14), generates useful identities.

For bosons, the eigenvalue expansion equation (18) applies likewise. The eigenvectors have the form given in equation (20), with ψ replaced by χ, and are normalized as in equation (16), with N replaced by N_B. The dimension $D_B^{(s)}$ of the subspace spanned by degenerate eigenvectors characterized by s with $s = 0, 1, \ldots, m$ is given by $D_B^{(0)} = 1$ and, for $s \geqslant 1$, by

$$D_B^{(s)} = \binom{l+s-1}{s}^2 - \binom{l+s-2}{s-1}^2. \tag{23}$$

Again, we have $\sum_{s=0}^{m} D_B^{(s)} = N_B^2$, showing that the eigenvectors form a complete set. The eigenvalues are given by

$$\Lambda_B^{(s)}(k) = \binom{m-s}{k}\binom{l+m+s-1}{k}. \tag{24}$$

The eigenvalues vanish for $s > m - k$, in keeping with the GUE result where for $k = m$ the only non-vanishing eigenvalue belongs to $s = 0$.

6.2. Group-theoretical aspects

Duality and the eigenvector decomposition equation (18) apply to both fermions and bosons. A group-theoretical analysis [35] shows that both these results and much of the structure displayed in subsection 6.1 apply much more widely and are rooted in symmetry properties of the embedded ensembles. The analysis also gives a deeper meaning to the concept 'embedded ensemble'. For simplicity, we confine ourselves to the unitary case although all results apply likewise to the orthogonal one. We denote by $\alpha_j^{\dagger}$ the creation operator for a fermion or a boson in single-particle state j and by $\Psi_{k\alpha}^{\dagger}$ the creation operator of a normalized state containing k particles (fermions or bosons). This normalized state need not be a Slater determinant or a

product state (as is the case for the operators $\psi_{k\alpha}$ and $\chi_{k\alpha}$ defined in section 2) but may be a linear combination of such states.

Three symmetry groups are relevant for the embedded ensembles. These are (i) the group $SU(l)$ of unitary transformations of the l degenerate single-particle states $|j\rangle$ with $j = 1, \ldots, l$; (ii) the group $U(N_k)$ of unitary transformations of the k-body interaction where N_k is the dimension of the Hilbert space containing k particles (fermions or bosons); (iii) the group $U(N_m)$ of unitary transformations of the Hilbert space containing m particles, with N_m the dimension of this space. As we shall see, the group $SU(l)$ governs the embedding, the group $U(N_k)$ is obviously the symmetry group of EGUE(k), and $U(N_m)$ is the symmetry group of the GUE.

An element $u \in SU(l)$ generates a unitary transformation of the single-particle states and a corresponding transformation $T(u)a_j^\dagger T^\dagger(u)$ of the creation operators $a_j^\dagger$. Under u, the operators $\Psi_{t\alpha}^\dagger$ transform according to the irreducible representation $D_{\alpha\gamma}^{f_t}(u)$,

$$T(u)\Psi_{t\alpha}^\dagger T^\dagger(u) = \sum_{\gamma} D_{\alpha\gamma}^{f_t}(u)\Psi_{t\gamma}^\dagger. \tag{25}$$

Here t is an integer which may take the values m, k or $(m - k)$. For fermions (bosons), the matrices D are in essence totally antisymmetrized (symmetrized, respectively) powers of u.

In analogy with the familiar fractional-parentage technique, we can expand the m-particle states $|m\alpha\rangle = \Psi_{m\alpha}^\dagger|0\rangle$ into products of states containing k and $(m - k)$ particles, respectively,

$$|m\alpha\rangle = \binom{m}{k}^{-1/2} \sum_{\gamma\delta} \Psi_{k\gamma}^\dagger |(m-k)\delta\rangle C_{f_k\gamma f_{(m-k)}\delta}^{f_m\alpha}. \tag{26}$$

The coefficients $C_{f_k\gamma f_{(m-k)}\delta}^{f_m\alpha}$ are the coefficients of fractional parentage or, equivalently, the Clebsch–Gordan coefficients for the coupling $[(f_k f_{(m-k)})f_m]$ of the irreducible representations f_k and $f_{(m-k)}$ to f_m.

The product $\Psi_{k\gamma}^\dagger\Psi_{k\delta}$ of operators which appears in the definition of the k-body embedded ensemble transforms according to the direct product of irreducible representations $D^{f_k}(u)$ and $D^{\overline{f_k}}(u)$. The product can be reduced to a direct sum of irreducible representations $D^{g_b}(u)$. This defines a set of basic k-particle interactions $B_k(b\alpha)$ which are Hermitean, transform according to the irreducible representation $D^{g_b}(u)$, and are given by vector-coupling $\Psi_{k\gamma}^\dagger$ and $\Psi_{k\delta}$ via the Clebsch–Gordan coefficient $C_{f_k\gamma\overline{f_k}\delta}^{g_b\alpha}$. Using the Wigner–Eckart theorem, we can write the matrix element $\langle m\gamma|B_k(b\alpha)|m\delta\rangle$ as the product of a reduced matrix element $\langle m||B_k(b)||m\rangle$ and the Clebsch–Gordan coefficient $C_{f_m\gamma\overline{f_m}\delta}^{g_b\alpha}$.

The interaction V_k can now be rewritten as a sum over the operators $B_k(b\alpha)$. The coefficients are uncorrelated Gaussian distributed random variables. Using this form and the Wigner–Eckart theorem, one finds for the second moment of the matrix elements $\langle m\gamma|V_k|m\delta\rangle$ an expression which is equivalent to the eigenvalue decomposition equation (18). The role of the eigenvectors is played now by the Clebsch–Gordan coefficients, and the role of the eigenvalues is taken by the squares of the reduced matrix elements $\langle m||B_k(b)||m\rangle$. The index b coincides with the summation index s in equation (18). The duality relation also follows from the Wigner–Eckart theorem. Thus, all the relations derived explicitly for fermions and bosons are seen to follow from group-theoretical considerations.

These insights allow for a generalization of the concept of an embedded ensemble which brings the underlying concepts to the fore most clearly. We consider an arbitrary compact simple Lie group G and two independent systems labelled k and $(m-k)$ whose basic states $|f_k\alpha\rangle$ and $|f_{(m-k)}\gamma\rangle$ transform according to the irreducible representations $D^{f_k}(g)$ and $D^{f_{(m-k)}}(g)$ of the group G, with $g \in G$. Now let us assume that a non-trivial interaction V_k of GUE type

occurs only in system k. The embedding of this interaction into a space of different dimension is accomplished by projecting the product states $|f_k\alpha\rangle|f_{(m-k)}\gamma\rangle$ onto the subspace of states which transform according to the irreducible representation $D^{f_m}(g)$ which is contained in the direct product $D^{f_k}(g) \times D^{f_{(m-k)}}(g)$. Let the associated projection operator be denoted by $I(f_m)$. The embedded ensemble is then defined by the Hamiltonian $I(f_m)V_k I(f_m)$. This definition comprises the essence of the group-theoretical extension of the idea of an embedded ensemble. It is independent of the existence of fermions and bosons and relies only on group-theoretical concepts.

Group-theoretical arguments can also be used to isolate that part of V_k which is invariant under $U(N_m)$, and to investigate symmetries of the generating functional in the supersymmetry approach [35]. The part of V_k which is invariant under $U(N_m)$ transforms either like the GUE or like a multiple of the unit operator. Both possibilities actually occur in the decomposition. The latter, taken by itself, would cause Poissonian level statistics. It carries particularly large weight for $k \ll m$.

6.3. Moments of V_k

The eigenvector expansion equation (18) and the orthogonality relations equation (16) make it possible to calculate low moments and variances of V_k without resorting to the dilute limit $l \gg m$, at least for $\beta = 2$. Three observables of interest are

$$S = \frac{\overline{((1/N)\,\mathrm{tr}V_k(\beta))^2}}{(1/N)\,\mathrm{tr}\overline{(V_k(\beta))^2}}$$

$$R = \frac{\overline{((1/N)\,\mathrm{tr}(V_k(\beta))^2)^2} - ((1/N)\,\mathrm{tr}\overline{(V_k(\beta))^2})^2}{((1/N)\,\mathrm{tr}\overline{(V_k(\beta))^2})^2} \tag{27}$$

$$\kappa = 2 + Q = \frac{(1/N)\,\mathrm{tr}\overline{(V_k(\beta))^4}}{((1/N)\,\mathrm{tr}\overline{(V_k(\beta))^2})^2}.$$

The ratio S measures the fluctuations of the centre of the spectrum in units of the average width of the spectrum. The ratio R measures the relative fluctuation of the width of the spectrum. The parameter κ is the kurtosis, and Q marks the difference in spectral shape between the semicircle ($Q = 0$) and the Gaussian ($Q = 1$). Explicit values for S, R, Q are given in [4, 2], both for fermions and for bosons. The result for R was first obtained by French [20]. Both S and R vanish in the limit $l \to \infty$. However, for fixed values for k and m/l, both ratios decrease very slowly (with inverse powers of the logarithm) with increasing dimension N of the matrices. This fact is at the root of the difficulty in obtaining reliable spectral information on EGE(k) from numerical simulations.

For the shape of the spectrum, the quantity Q is of primary interest. It is explicitly given [4] in terms of the eigenvalues $\Lambda^{(s)}(k)$ and the dimensionalities $D^{(s)}$ introduced above. In the limit $l \to \infty$, Q vanishes if $2k > m$ with both k and m fixed, and with both k/m and m/l fixed, while Q attains a finite and non-zero value for $2k \leqslant m$ with both k and m fixed, and with both k and m/l fixed. This suggests that the transition from the semicircular to Gaussian shape takes place for $2k \leqslant m$. (It would take the study of higher moments to make this conclusion unambiguous). As mentioned above, the critical role played by the value $2k = m$ is attributed to duality.

For bosons in the dense limit, the ratios S and R do not vanish as $m \to \infty$. Thus, the fluctuations of the centroids and widths of individual spectra do not vanish asymptotically. Hence, the bosonic ensembles are not ergodic in the dense limit $m \to \infty$ with k and l fixed.

6.4. Supersymmetry approach

The eigenvector expansion equation (18) makes it possible to apply the supersymmetry approach. To see this, we recall that when one uses supersymmetry to calculate properties of the GUE, equation (19) is extremely helpful. Indeed, after averaging the generating function over the ensemble, there appears in the exponent a term which is quartic in the commuting and anticommuting integration variables. Owing to equation (19), this quartic term can be written as the square of a bilinear form, and the latter is handled with the help of the Hubbard–Stratonovich (HS) transformation. Because of equation (18), a similar situation arises for EGE(k). The form of this equation implies that the quartic term in the integration variables obtained after ensemble-averaging, is a sum of squares of bilinear forms, and the HS transformation again can be used. There is a price, however: whereas only a single graded matrix σ is needed to carry out the HS transformation for the GUE, the EGE(k) requires the introduction of as many σ-fields as there are independent eigenvectors to non-zero eigenvalues in equation (18). This number rises steeply as m increases from $k = m$, see equation (21).

The saddle-point approximation to the integration over the σ-fields yields for EGE(k) a semicircle for the spectral density, and universal Wigner–Dyson spectral fluctuations, just as for the GUE. The loop expansion, generated by expanding the σ-fields around the saddle-point solution, serves as a test of the saddle-point approximation. For the GUE, all terms in the loop expansion vanish asymptotically for $N \rightarrow \infty$. For EGE(k), it is not possible technically to go beyond the lowest non-vanishing term of the loop expansion. For the spectral density, one finds that, for $2k > m$, this term vanishes asymptotically for $l \rightarrow \infty$. This fact reinforces the conclusion in subsection 6.3 that the spectral density has semicircular shape for $2k > m$. For $2k \leqslant m$, finite corrections appear, the form of which is consistent with the value of Q found in subsection 6.3. For the spectral fluctuations (the two-point function), the situation is more ambiguous. The lowest-order loop correction vanishes asymptotically for all values of k albeit it does so ever more slowly as k decreases from $k = m$. This might be consistent with universal Wigner–Dyson level statistics for all values of k. We note, however, that the lowest-order loop correction also produces non-universal terms. Such terms might also arise in higher order and not vanish asymptotically.

In summary, the supersymmetry approach yields useful information especially for $2k > m$. It lends independent support to the conclusion that the change from semicircular to Gaussian spectral shape sets in as k decreases and passes through the value $2k = m$. For the spectral fluctuations, it is consistent with but not necessarily in support of Wigner–Dyson statistics for all values of k. However, for the physically most interesting case $k = 2$, it yields very little information.

7. Binary correlation approximation

None of the analytical approaches described above has been able to yield definitive information on the spectral fluctuation properties of EGE(k) for the physically interesting case $k \ll m$. There exists, however, another approach first introduced by Mon and French for the calculation of the one-point function [32]. This is the binary correlation approximation which applies in the dilute limit $k \ll m \ll l$, $m/l \rightarrow 0$ as $m, l \rightarrow \infty$ [14]. The binary correlation approximation can be generalized to investigate the two-point function [3, 4]. We first review the calculation of the one-point function of the EGUE(k) in the version of Verbaarschot and Zirnbauer [37], and then present that of the two-point function [3, 4]. We note that in the dilute limit the distinction between fermions and bosons is physically irrelevant, both cases yielding the same

results. For definiteness, we consider the case of fermions. To be free of singularities, we choose in this section $v^2 = [\Lambda^{(0)}(k)]^{-1}$ which implies a bounded spectrum of unit width.

7.1. One-point function

Following Verbaarschot and Zirnbauer [37], we expand the one-point function in a power series in V_k,

$$\overline{g(z)} \equiv \frac{1}{N} \operatorname{tr}\left(\overline{\frac{1}{z - V_k}}\right) = \frac{1}{N} \sum_{p=0}^{\infty} \frac{1}{z^{2p+1}} \operatorname{tr}\left(\overline{V_k^{2p}}\right) \tag{28}$$

where $z = E - \mathrm{i}\eta$, E is the energy and $\eta > 0$ is an infinitesimal increment. In the second equation in (28), we have interchanged the summation and the average over the ensemble, and we have used the Gaussian distribution of V_k which implies that only even powers of V_k contribute. The ensemble average is carried out using Wick contractions. Each contracted pair of V_k is evaluated in the dilute limit, i.e. replaced by $v^2 \Lambda^{(0)}(k) = 1$. In the term of order p, there are $(2p - 1)!!$ different ways of pairwise contracting the V_k. The result is

$$\overline{g(z)} = \sum_{p=0}^{\infty} \frac{1}{z^{2p+1}} (2p - 1)!! = \frac{1}{z} \sum_{p=0}^{\infty} \frac{1}{p!} \left[\frac{1}{2z^2}\right]^p (2p)!. \tag{29}$$

This expression is evaluated using the technique of Borel resummation. We use the identity $n! = \int_0^{\infty} \mathrm{e}^{-t} t^n \, \mathrm{d}t$ for the factor $(2p)!$. We interchange the summation and the integration. The sum over p yields an exponential function, thus leading to

$$\overline{g(z)} = \frac{1}{z} \int_0^{\infty} \mathrm{d}t \exp\left[\frac{t^2}{2z^2} - t\right]. \tag{30}$$

Equation (30) converges provided $\operatorname{Re}[z^{-2}] < 0$. Writing $z = -\mathrm{i}|z|\mathrm{e}^{\mathrm{i}\phi}$, we see that this condition is fulfilled for $-\frac{\pi}{4} < \phi < \frac{\pi}{4}$. The integral is evaluated as follows. First, we choose as contour the straight line that joins the origin and z. Putting $t = \tau \mathrm{e}^{\mathrm{i}\phi}$, we rotate this contour so that it comes to lie on the real axis. A new (real) integration variable $u = \tau/|z|$ is introduced. The resulting integral converges for all z, and is evaluated in a straightforward manner, yielding [37]

$$\overline{g(z)} = \mathrm{i}\sqrt{\frac{\pi}{2}} \exp\left[-\frac{z^2}{2}\right] \operatorname{erfc}\left[\frac{\mathrm{i}z}{\sqrt{2}}\right]. \tag{31}$$

Taking the limit $\eta \to 0$ and recalling that the mean level density is given by $\overline{\rho(E - \mathrm{i}\eta)} = \pi^{-1} \operatorname{Im}[\overline{g(E - \mathrm{i}\eta)}]$ we obtain the Gaussian form for the average level density

$$\overline{\rho(E)} = (2\pi)^{-1/2} \exp\left[-\frac{E^2}{2}\right]. \tag{32}$$

7.2. Two-point function

Our presentation differs slightly from that in [4] and takes account of recent developments [36]. Again, we work in the dilute limit and consider the quantity

$$R_2(z_1, z_2) = \frac{\overline{g(z_1)g(z_2)}}{\overline{g(z_1)} \cdot \overline{g(z_2)}} - 1 \tag{33}$$

where $z_1^{\pm} = E^{\pm} + \varepsilon/2$ and $z_2^{\pm} = E^{\pm} - \varepsilon/2$, with E and ε the mean value and the difference of the energy arguments of the two Green functions, respectively. An upper plus (minus) sign

denotes an infinitesimal positive (negative) imaginary energy increment, respectively. We are interested in values of ε which are of the order of the mean level spacing and, thus, small compared to unity (the width of the spectrum). Thus, we have approximately $|z_1| \sim |z_2|$. In order to obtain the connected part of the density–density correlator, z_1 and z_2 must have imaginary energy increments of opposite signs.

Proceeding as in the case of the one-point function, we expand $R_2(z_1, z_2)$ in powers of V_k. Using equation (28) we have

$$\overline{g(z_1)g(z_2)} = \frac{1}{N^2} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{1}{z_1^{p+1} z_2^{q+1}} \overline{\text{tr}[(V_k)^p]\, \text{tr}[(V_k)^q]}. \tag{34}$$

The Gaussian distribution of the V_k implies that $p + q$ must be even. Using Wick contraction we obtain two types of contributions. The members of the contracted pair either occur under the same trace, or involve both traces. We refer to the latter as cross-contractions. Reordering terms, we obtain

$$\overline{g(z_1)g(z_2)} = \frac{1}{N^2} \sum_{s=0}^{\infty} \sum_{p,q=0}^{\infty} \frac{1}{z_1^{2p+s+1} z_2^{2q+s+1}} \overline{(\text{tr}[(V_k)^{2p+s}]\, \text{tr}[(V_k)^{2q+s}])_s}. \tag{35}$$

In equation (35), the index s counts the number of cross-contracted pairs. There are $\binom{2p+s}{s}\binom{2q+s}{s}$ different ways of cross-contracting the V_k, while there are $(2p-1)!!(2q-1)!!$ ways of pairwise contracting the remaining V_k. For the latter, we use the binary correlation approximation. Thus, each one of these contractions yields a factor unity, irrespective of the position where the contracted V_k appear in each of the traces. The result is

$$\overline{g(z_1)g(z_2)} = \frac{1}{N^2} \sum_{s=0}^{\infty} \sum_{p,q=0}^{\infty} \frac{(2p-1)!!(2q+1)!!}{z_1^{2p+s+1} z_2^{2q+s+1}} \binom{2p+s}{s}\binom{2q+s}{s} \overline{(\text{tr}[(V_k)^s]\, \text{tr}[(V_k)^s])_s}. \tag{36}$$

In [4], it was argued that in the limit $l \to \infty$ the terms with $s \neq 0$ become negligibly small in comparison with the terms with $s = 0$. The latter correspond to the unlinked contributions. Upon resummation, these terms factorize into the product $\overline{g(z_1)} \cdot \overline{g(z_2)}$. Therefore, the two-point correlation function $R_2(z_1, z_2)$ defined in equation (28) vanishes asymptotically in the limit $l \to \infty$ and, thus, implies a Poissonian spectrum. In detail, the argument is as follows.

Observing that the binomial factors and the powers of z_1 and z_2 in equation (36) do not depend on l or on m, and recalling that $|z_1| \sim |z_2|$, we see that the only l-dependent factor is

$$T_s = \frac{1}{N^2} \overline{(\text{tr}[(V_k)^s]\, \text{tr}[(V_k)^s])_s}. \tag{37}$$

Obviously $T_0 = 1$ and, thus, all unlinked contributions are constant and independent of l. Using counting arguments, the dilute limit, and Stirling's formula, one obtains the asymptotic estimate $T_s \sim l^{-sk}$ for T_s [4]. This result implies that all connected contributions vanish in the limit $l \to \infty$ at least as l^{-k}. We note that in the dilute limit, l^{-k} vanishes as a power of the logarithm of the Hilbert space dimension N. Hence the two-point function $R_2(z_1, z_2)$ vanishes. This line of reasoning can also be applied to the n-point correlation function with $n > 2$ showing that the spectrum is Poissonian.

This argument has the obvious flaw that it does not display the role of ε, the difference of the energy arguments of the two Green functions. By the same token, the argument does not display the difference between the connected part of the two-point function (where z_1 and z_2 carry infinitesimal imaginary increments of opposite signs) and the product of two advanced (or two retarded) Green functions (where z_1 and z_2 carry infinitesimal imaginary increments

of the same sign). In the case of canonical RMT, we know that in the first case (z_1 and z_2 carry infinitesimal imaginary increments of opposite signs) the analogue of R_2 does not vanish asymptotically (in fact, the asymptotic value of this function determines the Δ_3-statistic and the stiffness of the spectrum) while in the second case (z_1 and z_2 carry infinitesimal imaginary increments of the same sign) the analogue of R_2 vanishes asymptotically. This implies that in the case of canonical RMT, the analogue of R_2 has a discontinuity across the real energy axis. It is to be expected that such a discontinuity exists likewise for $R_2(z_1, z_2)$.

Srednicki [36] has recently drawn attention to this singular behaviour and has considered the convergence properties of the sums in equation (35) in the light of this question. Equation (36) is symmetric in the indices p and q. This suggests writing $R_2(z_1, z_2)$ as

$$R_2(z_1, z_2) = \sum_{s=1}^{\infty} \frac{g_s(z_1) g_s(z_2)}{g(z_1) \cdot g(z_2)} T_s \tag{38}$$

where the function $g_s(z)$ is given by

$$g_s(z) = \sum_{p=0}^{\infty} \frac{(2p-1)!!}{z^{2p+s+1}} \binom{2p+s}{s} = \frac{1}{s! z^{s+1}} \sum_{p=0}^{\infty} \frac{(2p+s)!}{p!(2z^2)^p}. \tag{39}$$

In equation (38) we have used the fact that $g_{s=0}(z) = \overline{g(z)}$, see equation (29). Using Borel resummation and working out the resulting integrals as in the case of the one-point function, one is led to the integral representation [36]

$$g_s(z^{\pm}) = \frac{(\mp i)^{s+1}}{s!} \int_0^{\infty} du\, u^s \exp\left[-\frac{u^2}{2} \pm i z^{\pm} u\right]. \tag{40}$$

The upper plus (minus) sign refers to a positive (negative) infinitesimal imaginary energy increment, respectively. Using equation (40) in equation (38) yields

$$R\left(z_1^+, z_2^{\pm}\right) = \frac{\mp 1}{g\left(z_1^+\right) \cdot g\left(z_2^{\pm}\right)} \int_0^{\infty} du\, dv\, e^{-(u^2+v^2)/2}\, e^{i(z_1^+ u \pm z_2^{\pm} v)} F(\mp uv) \tag{41}$$

where the function $F(y)$ is defined as

$$F(y) = \sum_{s=1}^{\infty} \frac{y^s}{(s!)^2} T_s. \tag{42}$$

Srednicki points out that, in order for both $R_2\left(z_1^+, z_2^+\right)$ and $R_2\left(z_1^+, z_2^-\right)$ to be well defined, $F(y)$ must be free of singularities on the real axis, in which case the integrals can be performed 'without further (arbitrary) regularization'. Using the case $k = 1$, he shows that $F(y)$ does have a singularity. This then invalidates the conclusion drawn in [4] that the spectrum is Poissonian. Srednicki concludes that the spectral statistics of the EGE(k) in the dilute limit remains an unsolved problem [36]. The example $k = 1$ considered by Srednicki is all the more remarkable because, in this case, it is clear from independent arguments that the spectrum is Poissonian for $m \gg 1$.

8. Correlated matrix elements: the limiting ensembles

Differences between the spectral fluctuations of EGE(k) and canonical RMT may be attributed to the fact that in EGE(k), the many-body matrix elements are strongly correlated while in RMT they are not. The correlations are apparent from the definition of the ensemble, equation (4): matrix elements not related by symmetry which involve different many-body states $\psi_{k,\alpha}$, may have the same value. It is instructive to study whether and how these correlations of

the matrix elements influence the spectral statistics. We do so by maintaining the *graphical structure* of the EGE(k). By this we mean that many-body matrix elements which vanish for the EGE(k) continue to have the value zero. We modify only the values of the non-vanishing matrix elements in such a way as to display the role of the correlations. It is in this spirit that we construct two limiting ensembles, one with maximum and the other with minimum correlations among the matrix elements. Both ensembles have the same graphical structure as the embedded ensembles. The EGE(k) will be seen to lie between these two limiting ensembles. For simplicity we consider only fermions. We essentially follow [4].

The graphical structure of the embedded ensembles may be displayed by assigning to each Hilbert space vector $\psi_{k,\alpha}$ a vertex α, and to each non-diagonal matrix element $\langle\mu|V_k|\nu\rangle$ which is not identically equal to zero, a link connecting the vertices μ and ν. The diagonal matrix elements $\langle\mu|V_k|\mu\rangle$ are represented by loops attached to the vertex μ. With the exception of the loops, the resulting structure is referred to as a 'regular graph' in the mathematical literature [31]. (For bosons, the resulting structure is not a regular graph.) The number of vertices is obviously given by N, the dimension of Hilbert space. For fermions, the number M of links emanating from any given vertex is the same for all vertices and given by

$$M = \sum_{s=1}^{k} \binom{m}{s}\binom{l-m}{s}.$$ (43)

This result is obtained by counting the non-diagonal matrix elements $\langle\mu|V_k|\nu\rangle$ that connect states which differ in the occupation numbers of at most k particles. For $k < m$, we have $M < N - 1$ while $M = N - 1$ for $k = m$. The number P of independent links is given by the number of matrix elements above the main diagonal which do not vanish identically. Hence,

$$P = \frac{1}{2}MN.$$ (44)

Obviously, N, M and P do not depend on the symmetry parameter β.

To obtain a measure for the correlations between matrix elements, we define the number K_β of independent random variables. This number differs for the unitary and the orthogonal ensembles and is given by

$$K_\beta = \frac{\beta}{2}\binom{l}{k}\left[\binom{l}{k} + \delta_{\beta 1}\right].$$ (45)

The ratio K_β/P of the number K_β of independent random variables and the number P of different links serves as a measure of the correlations. For $k \ll m$, K_β/P is much smaller than one. It approaches the value $\beta(N + \delta_{\beta 1})/(N - 1)$ monotonically from below as k approaches m. This shows that for $k \ll m$ there are strong correlations between matrix elements on different links. The correlations disappear as we approach the canonical limit $k = m$. This is illustrated for several parameters k, m and l in figure 4.

We now define the two limiting ensembles by assigning a minimum and a maximum number of independent random variables to the graph structure of the embedded ensembles. For the EGE(k), the parameter K_β/P lies between the values associated with these limiting ensembles. Obviously, the minimum number of random variables is one. The corresponding ensemble EGE$_{\min}(k)$ is defined in terms of the matrix elements $\langle\mu|V_k^{\min}|\nu\rangle$ of the operator

$$V_k^{\min} = v\sum_{\alpha\gamma}\psi_{k,\alpha}^{\dagger}\psi_{k,\gamma}.$$ (46)

The factor v is a real (complex) Gaussian random variable for $\beta = 1$ ($\beta = 2$, respectively). Without loss of generality we may, however, put $\overline{|v|^2} = 1$, removing the distinction between the unitary and the orthogonal cases. The ensemble EGE$_{\max}(k)$ containing the maximum number
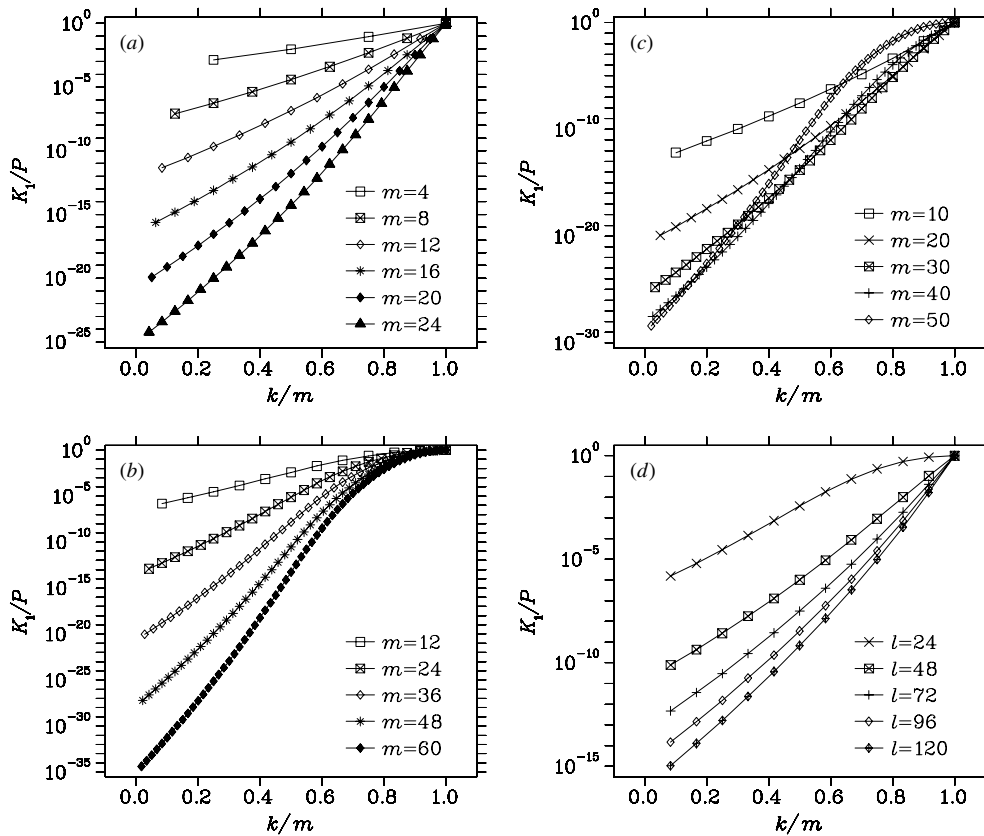
Figure 4. The ratio K_1/P, see equations (44) and (45), on a logarithmic scale versus k/m. Panel (*a*): $f = m/l = 1/5$. Panel (*b*): $f = m/l = 1/2$. Panel (*c*): $l = 100$. Panel (*d*): $m = 12$. Taken from [4].

of independent random variables is obtained by assigning, within the constraints imposed by symmetry, a different random variable $v_{\mu\nu}$ to each link of the graph. The matrix elements of $\text{EGE}_{\text{max}}(k)$ are given by

$$\langle\mu|V_k^{\text{max}}|\nu\rangle = v_{\mu\nu}\sum_{\alpha\gamma}\langle\mu|\psi_{k,\alpha}^{\dagger}\psi_{k,\gamma}|\nu\rangle. \tag{47}$$

For $\beta = 1$ ($\beta = 2$), the matrix $v_{\mu\nu}$ is real symmetric (complex Hermitean, respectively). Elements not connected by symmetry are uncorrelated Gaussian random variables with mean value zero and variance $\overline{v_{\mu\nu}v_{\mu'\nu'}} = \delta_{\mu\nu'}\delta_{\nu\mu'} + \delta_{\beta1}\delta_{\mu\mu'}\delta_{\nu\nu'}$.

The ensemble $\text{EGE}_{\text{min}}(k)$ is fully integrable and has spectral fluctuations which are not of Wigner–Dyson type. For $k = 1$, one finds two different degenerate eigenvalues $\lambda_1 = vl$ and $\lambda_2 = 0$, with degeneracies $n_1 = \binom{l-1}{m-1}$ and $n_2 = \binom{l-1}{m}$. For $k = m$, the matrix representation of $\text{EGE}_{\text{min}}(m)$ in Hilbert space carries the entry v on every element. Diagonalization of this matrix is trivial and gives the eigenvalues Nv (non-degenerate) and zero ($(N-1)$-fold degenerate). Again, the ensemble is fully integrable, and the spectral fluctuations are not Wigner–Dyson. Using the supersymmetry method it can be shown that the spectral fluctuation properties of $\text{EGE}_{\text{max}}(k)$ coincide with the predictions of RMT. From these facts, it is apparent that the limiting ensembles cover the extreme cases of a fully integrable system and a system with

Wigner–Dyson spectral statistics. Figure 4 then suggests that as k increases, the spectral fluctuations of both EGOE(k) and EGUE(k) may undergo a gradual transition from Poissonian to Wigner–Dyson behaviour.

9. Conclusions

We return to the questions raised at the end of section 1. We recall that both for $k = m$ and in the dilute limit, fermions and bosons behave qualitatively similarly, while the dense limit for bosons is a special case.

(i) What is the shape of the spectral density? Among the four questions formulated in the introduction, this is the one to which we have a nearly complete answer. The moments method shows that there is a gradual transition from semicircular shape for $k = m$ to Gaussian shape in the dilute limit. This is supported by the binary correlation approximation which yields a Gaussian shape in the dense limit. The eigenvalue expansion of the matrix of second moments has added the insight that the semicircle prevails as long as $2k > m$, and that the transition to Gaussian shape sets in at $2k = m$. The special role of the value $2k = m$ is due to duality. The case of dense bosons is special: it is not ergodic.

(ii) What are the spectral fluctuation properties? The supersymmetry method suggests that the spectral fluctuations are of Wigner–Dyson type as long as $2k > m$. It is possible that the range of validity of Wigner–Dyson spectral statistics extends into the domain $2k \leqslant m$ although here the supersymmetry method yields also non-universal contributions. In the dilute limit, the situation is not clear. The straightforward extension of the binary correlation approximation yields Poissonian statistics for $l \rightarrow \infty$ but relies on manipulations which are mathematically questionable. The eigenvector expansion of the matrix of second moments yields expressions which change smoothly with k, m, l. From these expressions one would not expect a sudden transition from Poissonian statistics (which surely applies for $k = 1$) to Wigner–Dyson statistics for $k = 2$. The group-theoretical approach shows that the part of V_k which transforms under $U(N_m)$ like a multiple of the unit matrix (and, thus, leads to Poissonian level statistics) is largest for $k \ll m$. The study of the correlations between many-body matrix elements provides intuitive insight into the causes responsible for deviations from Wigner–Dyson statistics. Dense boson systems are close to integrable and display no similarity to Wigner–Dyson spectral statistics.

(iii) Are these properties universal? In analogy with canonical RMT, we ask here whether the spectral density and the spectral fluctuation properties hold irrespective of the Gaussian choice for the k-body matrix elements, i.e. the variables $v_{k;\alpha\gamma}$ defined in section 2. This is a meaningful question: we recall that in canonical RMT, ensembles with non-Gaussian weight factors have been studied. Since for $m = k$, EGE(m) is identical to canonical RMT, non-Gaussian ensembles of EGE(k) type can certainly be defined. The analytical methods reviewed in this paper all rely heavily on the Gaussian distribution of the $v_{k;\alpha\gamma}$. At this point in time, no analytical approach is in sight to answer this question, nor are we aware of any numerical simulations addressing the issue. Another aspect of universality is addressed in the group-theoretical analysis of the embedded ensembles. It shows that the embedded ensembles for fermions and bosons are part of a much wider class of embedded random-matrix ensembles defined in terms of irreducible representations of some compact Lie group.

(iv) Are the spectra ergodic? Partial answers to this question come from numerical simulations, from the moments method, and from the eigenvalue expansion of the matrix of second

moments. Ergodicity can be expected only in the limit of infinite matrix dimension. For $k = m$ ergodicity holds for all observables. In the dilute limit, the spectral density is ergodic, but non-ergodic contributions vanish very slowly (with inverse powers of the logarithm of the dimension of the matrices). This causes difficulties in the analysis of numerically simulated spectra. For dense bosons, even the low moments of the spectral density are not ergodic. It seems that this is the first known case of a random-matrix model which is not ergodic in the limit of infinite matrix dimension. We are not aware of any studies addressing ergodicity of the spectral fluctuations.

This review shows that after many years of work, the determination of the spectral density and, especially, of the spectral fluctuation properties of EGE(k) still poses difficult problems. The difficulties are due to the fact that EGE(k) lacks the symmetry properties of the canonical ensembles of RMT.

Acknowledgments

References

[1] Asaga T, Benet L, Rupp T and Weidenmüller H A 2001 *Europhys. Lett.* **56** 340
[2] Asaga T, Benet L, Rupp T and Weidenmüller H A 2002 *Ann. Phys., NY* **298** 229
[3] Benet L, Rupp T and Weidenmüller H A 2001 *Phys. Rev. Lett.* **87** 010601-1
[4] Benet L, Rupp T and Weidenmüller H A 2001 *Ann. Phys., NY* **292** 67
[5] Benet L, Bohigas O, Jung C and Leyvraz F in preparation
[6] Bijker R, Frank A and Pittel S 1999 *Phys. Rev.* C **60** 021302R
[7] Bijker R and Frank A 2000 *Phys. Rev. Lett.* **84** 420
[8] Bijker R and Frank A 2000 *Phys. Rev.* C **62** 014303
[9] Bohigas O and Flores J 1971 *Phys. Lett.* B **34** 261
[10] Bohigas O and Flores J 1971 *Phys. Lett.* B **35** 383
[11] Bohigas O and Flores J 1972 *Statistical Properties of Nuclei* ed J B Garg (New York: Plenum) 195
[12] Bohigas O, Flores J, French J B, Giannoni M J, Mello P A and Wong S S M 1974 *Phys. Rev* C **10** 1551
[13] Bohigas O and Giannoni M J 1975 *Ann. Phys., NY* **89** 393
[14] Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S M 1981 *Rev. Mod. Phys.* **53** 385
[15] Cota E, Flores J, Mello P A and Yépez E 1974 *Phys. Lett.* B **53** 32
[16] Dyson F J 1962 *J. Math. Phys.* **3** 140
[17] Flores J, Horoi M, Müller M and Seligman T H 2001 *Phys. Rev.* E **63** 026204
[18] French J B and Wong S S M 1970 *Phys. Lett.* B **33** 449
[19] French J B and Wong S S M 1971 *Phys. Lett.* B **35** 5
[20] French J B 1973 *Rev. Mex. Fís.* **22** 221
[21] Gervois A 1972 *Nucl. Phys.* A **184** 507
[22] Guhr T, Müller-Groeling A and Weidenmüller H A 1998 *Phys. Rep.* **299** 189
[23] Jackson A D, Mejía-Monasterio C, Rupp T, Saltzer M and Wilke T 2001 *Nucl. Phys.* A **687** 405
[24] Johnson C W, Bertsch G F and Dean D J 1998 *Phys. Rev. Lett.* **80** 2749
[25] Johnson C W, Bertsch G F, Dean D J and Talmi I 1999 *Phys. Rev.* C **61** 014311
[26] Kaplan L and Papenbrock T 2000 *Phys. Rev. Lett.* **84** 4553
[27] Kota V K B and Potbhare V 1980 *Phys. Rev.* C **21** 2637
[28] Kota V K B 2001 *Phys. Rep.* **347** 223
[29] Laberge G J H and Haq R U 1990 *Can. J. Phys.* **68** 301
[30] Manfredi V R 1984 *Lett. Nuovo Cimento* **40** 135
[31] Marshall C W 1971 *Applied Graph Theory* (New York: Wiley–Interscience)
[32] Mon K K and French J B 1975 *Ann. Phys., NY* **95** 90
[33] Papenbrock T, Kaplan L and Bertsch G F 2002 *Phys. Rev.* B **65** 235120

[34] Patel K, Desai M S, Potbhare V and Kota V K B 2000 *Phys. Lett.* A **275** 329
[35] Pluhar Z and Weidenmüller H A 2002 *Ann. Phys., NY* **297** 344
[36] Srednicki M 2002 Spectral statistics of the *k*-body random-interaction model *Preprint* cond-mat/0207201
[37] Verbaarschot J J M and Zirnbauer M R 1984 *Ann. Phys., NY* **158** 78
[38] Wigner E P 1951 *Ann. Math.* **53** 36
[39] Wong S S M and French J B 1972 *Nucl. Phys.* A **198** 188